"WHAT DO YOU BELIEVE IS TRUE?" A ROUTINE FOR PROVING THEOREMS IN SECONDARY GEOMETRY

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We report findings from an investigation of one teacher's instruction as he guided students through the proofs of 21 theorems in a Grade 8 Honors Geometry course. We describe a routine involving four distinct phases, including Setting up the Proof and Concluding the Proof. Results from an end-of-course proof test are also presented to attest to the effectiveness of the teacher's approach. By engaging with descriptions of the theorem-proving routine, one can learn about different strategies that may support students to learn to prove theorems, such as asking students to put forth claims in the form of conjectures or other statements that they believe are true and seeking justifications for these claims as well as sanctioning a theorem once proven.

Keywords: Reasoning and Proof, Geometry and Spatial Reasoning

Purpose of the Study

In 1994, Alan Schoenfeld noted: "Proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics" (p. 76). Yet, despite the fact that reasoning and proving are considered central to the discipline of mathematics, and geometry is typically the place in the school mathematics curriculum where proof is taught, the teaching of proof in school geometry has been considered to be a failure in almost all countries (Balacheff, 1988). Acknowledging this failure, Battista (2007) posed the question: "How can proof skills best be developed in students?" (p. 888). To address this question, in this mixed-methods study, we focus on the development of proof skills with respect to the "instructional situation" (Herbst, Nachlieli, & Chazan, 2011) of proving theorems. Proving theorems is an activity that differs from doing proofs of "configurations" (Herbst & Miyakawa, 2008, p. 470) whereby students are typically provided with "Given" information, a "Prove" statement, and a figure to go along with the proof. Research conducted by Otten and colleagues (Otten, Gilbertson, Males, & Clark, 2014; Otten, Males, & Gilbertson, 2014) suggested that U.S. textbooks primarily engage students in proving configurations rather than theorems. This is a problematic situation if one agrees with Schoenfeld's (1994) argument that proof is an essential component of doing mathematics.

Because we agree with Reid and Knipping (2010) who suggested that recommended changes to how teachers teach proof must be based on detailed understandings of how teachers currently teach proof, we designed a study that involved spending significant time in teachers' classrooms. After determining that the students of one of the teachers in the study, who we call Shane, were outperforming other teachers' students in seemingly similar courses on an end-of-course proof test, we observed 22 of Shane's lessons during the 2018-2019 academic year. For this paper, we posed the following research question: *How did a teacher whose students were overall* *"successful" on an end-of-course proof test teach his students to prove theorems in geometry?* We operationalize what is meant by "successful" in subsequent sections of this paper.

Theoretical Framework

In order to frame the purpose of the study, justify the study methodology, and focus and guide the reporting and discussion of the results (Cai et al., 2019), we review three areas of literature. We first describe past results from the end-of-course assessment used in this study. Next, we describe research on "doing proofs" in U.S. geometry classrooms to support the need for the study. Last, we highlight research-based competencies for proving to frame the findings.

Students' Past Performance on an End-of-Course Proof Test

In Senk's (1985) paper titled, *"How Well Do Students Write Geometry Proofs?"* Senk described some of her research instruments and summarized some key findings from her (1983) dissertation. Senk's (1985) research question was: To what extent do secondary school geometry students in the United States write proofs similar to the theorems or exercises in commonly used geometry texts? Her results were part of the larger Cognitive Development and Achievement in Secondary School Geometry (CDASSG) Project. To answer her research question, Senk administered three forms of the CDASSG end-of-course proof test. Each form contained six items. The first item required students to fill-in-the-blanks of a two-column proof. The second item required a translation from a verbal statement to an appropriate "figure," "given," and "to prove." The last four items required students to write full proofs (Senk, 1985). Senk administered the CDASSG assessments to 1520 students in 74 classes from 11 schools in five states in 1981. Each item was scored on a four-point scale, and students were considered "successful" if they scored at least 3 out of 4 possible points. Students scored a 3 if their proof steps followed logically from previous ones but contained minor errors. Overall, Senk (1985) concluded that only about 30% of students in the full-year geometry courses that covered proof reached a 75% mastery of proof (i.e., were "successful" on the test overall). Senk also concluded that proofs of textbook theorems were difficult for many students. For example, only 32% of students were successful in proving the theorem that the diagonals of a rectangle are congruent, and 34% of students scored a 0 on this proof. A common error was citing the theorem in the proof (i.e., using circular reasoning). Across the three forms, an average of approximately 13% of students were successful on all six tasks with only 3% of students receiving perfect scores on all six.

"Doing Proofs" in U.S. School Geometry

Building on the past work of Lampert (1993) and Schoenfeld (1986, 1988, 1989) who documented the role that proof has traditionally played in classroom teaching and learning, Herbst and colleagues examined both students' and teachers' perspectives on what "doing proofs" is like in American high school geometry classrooms. Herbst and Brach (2006) reported findings from 29 interviews with 16 students in five categories: Statements, Initial Conditions, Concepts, Targets of the Work of Proving, and The Work of Proving. Several findings from Herbst and Brach's study are relevant to this study, especially students' claims that:

- It is customary that the "given" and prove" will be specified in the problem statement,
- Students are rarely asked to prove theorems, and
- The first thing in proving is to mark the givens on the diagram.

Herbst's and colleagues' (2009) study of teachers' views outlined a set of 25 norms for the instructional situation of "doing proofs," including the following norms about the division of labor - the teacher or textbook is responsible for:

- Establishing the givens in terms of properties of a figure represented in a diagram, and
- Providing a diagram that fairly represents the objects to be used in the proof.

The 25th norm was that every single statement or reason is produced in a handful of seconds. Overall, this research demonstrates that when "doing proofs" it is the teacher, not the students, who seems to carry much of the cognitive load.

Developing Competencies for Proving

Cirillo and colleagues' research has focused on understanding the conditions in which teachers currently teach proof in geometry with the ultimate goal of improving the teaching and learning of proof (see, e.g., Cirillo & Hummer, 2019). After observing that the classroom teachers with whom she worked were unsure about how to teach proof and were particularly unclear about how to *begin* teaching proof, Cirillo et al. (2017) developed a pedagogical framework for teaching proof based on the research literature and classroom observations. The pedagogical framework decomposes proof so that understanding of the larger goal (i.e., doing proofs) can be built up sequentially by teaching particular sub-goals of proof over time. The sub-goals of proof included in the framework that are particularly relevant to this study include: Knowing Geometric Concepts, Conjecturing, Working with Diagrams, Drawing Conclusions, Understanding Theorems, and Understanding the Nature of Proof. Particularly relevant competencies, which are nested within the sub-goals, include: Being able to turn a conjecture into a testable conditional statement; knowing how to read a diagram and understanding what can and cannot be assumed from a diagram; using axioms, postulates, definitions, and theorems to draw valid conclusions from some "Given" information; and being able to identify the hypothesis and conclusion of a conditional statement and then writing particular "Given and "Prove" statements, typically making use of a generic figure (see the full framework in Cirillo & May, 2021). Many of these competencies were also observed in Cirillo and Hummer's (2021) smartpen interview study in the work of students who were "successful" in completing proofs during the clinical interviews. For example, the following competencies were observed in the work of students who were successful with the proofs - students: productively attended to the "Given" information; used the diagram as a resource; identified warrants as postulates, axioms, definitions, or theorems; and attended to important details in their proofs.

Methods

In this paper we share results from a sub-study of a larger study focused on improving the teaching and learning of proof in secondary geometry. The larger project, *Proof in Secondary Classrooms* (Cirillo, 2015-2020), is a mixed methods study that took place in the mid-Atlantic region of the United States. Here, we focus on a subset of participants from the larger study who did not receive the study treatment (i.e., they were in the control group).

Context and Data

Across the three years of the sub-study, with the help of the research project staff, six teachers who taught a total of 464 Grade 8 Honors Geometry students administered Senk's (1983) CDASSG assessment at the end of the school year. It is important to note that prior to adopting the CDASSG for our study, through an alignment analysis, we concluded that the

CDASSG was, in fact, well aligned with current standards and textbooks being used in the study classrooms. The assessments were scored, and results were analyzed each summer. Beginning in Year 1 of the test administration, we noticed that, in comparison to the other sections of Grade 8 Honors students, one teacher's students consistently scored higher on the CDASSG assessment. More specifically, we noticed that in Year 1, the students (n=43) of the teacher, who we call Shane, earned a mean score of 19.05 out of 24 possible points on the six-item proof assessment (i.e., 79%); whereas the Grade 8 Honors Geometry students (n=129) in other teachers' classes earned a mean score of 8.5 out of 24 (i.e., 35%). Upon noticing this, we became interested in observing Shane's teaching, and we asked to observe his proof-focused lessons. Consequently, we conducted 22 classroom observations in one section of Shane's Grade 8 Honors Geometry course during the 2018-2019 school year. We requested that Shane invite us in when he first introduced proof up until and including lessons focused on quadrilateral proofs.

Qualitative Data Analysis

Phase 1: Identifying a reduced data set. The research team initially watched and developed timelines of the 22 classroom observation videos. These timelines identified the portions of the class that were dedicated to various classroom activities such as whole-class work, seatwork, and going over homework; within each activity, researchers included descriptions of the content covered. Of the 22 observations, we identified 11 observations where theorems were proved in the whole-class setting. Within these 11 observations, a total of 21 theorems were proved, comprising of approximately 6 hours and 23 minutes of video data. Transana Multiuser 3.32d (Woods, 2020) was used to transcribe and create a collection of video clips of each theorem-proof (i.e., proofs of actual theorems rather than "configurations" (Herbst & Miyakawa, 2008) including the Pre-Proof activities). The video clip collection was then further analyzed.

Phase 2: Identifying themes. The research team watched all 21 video clips of the whole-class theorem-proofs, as well as any related activities conducted prior to the theorem-proof (i.e., pre-proof activities) and looked for patterns within these data. We identified three distinct activities that occurred during the teaching of theorem-proofs: Setting up the Proof, Making and Justifying Claims, and Concluding the Proof.

Phase 3: Coding the themes. We developed codes to further analyze the three activities. Codebooks for each activity were developed using constant comparative analysis (Boeije, 2002). The codebooks were continuously revised and improved as each activity was coded in teams of two. Each pair of researchers independently coded at least 3 of the 21 theorem-proofs for their specific activity. After achieving above 80% interrater agreement (i.e., 90% for Setting Up the Proof, 92% for Making and Justifying Claims, and 92% for Concluding the Proof) and reconciling differences, coders worked independently to code the remaining data.

Quantitative Data Analysis

To provide further information about Shane's students' performance on the end-of-course CDASSG proof assessment in comparison to other similar students' performance on the same assessment, we analyzed results from two particular items of the CDASSG. More specifically, we focused on results from Senk's CDASSG Items 4 and 5. These two items were selected because across all three forms of the CDASSG, the items were similar in nature and explicitly required students to write full proofs. In particular, each form of the test included a proof of a theorem for one of the two items (e.g., the measures of the angles of a triangle sum to 180° or the diagonals of a rectangle are congruent), and the second item was a configuration proof involving triangle congruence. Following Senk, we report the percentage of students who were "Successful" and "Not Successful" on these items, where "Successful" means that students scored at least 3 out of 4 points on the item. Results across three years of the study are shared for Shane's Grade 8 Honors Geometry students and for all other Grade 8 Honors Geometry students in the study who were also in the control group (i.e., did not receive the project treatment).

Results

We share findings from five related analyses. We begin by exploring the four parts of Shane's theorem-proving routine: Pre-Proof: Making and Justifying Claims; Setting up the Proof; During-the-Proof: Making and Justifying Claims; and Concluding the Proof (see Figure 1). We then share additional quantitative data from the Grade 8 Honors Geometry student assessment results. This last finding is included to provide evidence of the effectiveness of Shane's approach to teaching proof. We begin by describing the two Making and Justifying Claims activities since they are closely related to one another.

Figure 1: Shane's Routine for Proving Theorems

Making and Justifying Claims in Pre-Proof and During-the-Proof Activities

We considered student-generated claims and justifications that were made both during "Pre-Proof" activities, which preceded Setting up the Proof, as well as "During-the-Proof," which followed Setting up the Proof. We only considered claims and justifications that were made by the students, rather than the teacher. Claims that were truly generated by the students without teacher support were made 44% of the time, and claims that were generated by the students as a result of question-and-answer exchanges between Shane and the students occurred 56% of the time. Throughout all 21 whole-class discussions of the theorem-proofs, Shane used the word "believe" 130 times, asking questions such as: "What do you *believe* is true?," "Do you *believe* it's always true?," and "Do you have a reason for why you *believe* that?"

Pre-Proof Claims and Justifications. Pre-Proof activities included exploring definitions to better understand the geometric objects involved in the proof (e.g., developing or stating definitions of isosceles triangles or parallelograms) and making claims that were sometimes unsupported and considered to be conjectures or were valid conclusions that could be drawn from a proof assumption. Across the 21 theorems, we identified 28 claims made during the Pre-Proof activities. Three of these claims were related to establishing a definition of the geometric object centrally involved in the proof. Fifteen of the claims were conjectures that would ultimately be considered "worth proving;" that is, the students conjectured the proof idea through a discovery process led by Shane prior to the Setting-Up-the-Proof activity that followed. Two of the 28 claims were generated through a combination of some assumption that could be made about a diagram and a postulate (e.g., AB + BC = AC by the Line Segment Addition Postulate). The remaining eight claims were conclusions that could be drawn from the premise of the proof. For example, if Shane presented some quadrilateral ABCD that was assumed to be a parallelogram (i.e., eventually the proof hypothesis or "Given" statement), then students would state a valid claim that the two pairs of opposite sides of the quadrilateral were parallel. The justification for this claim would be the definition of parallelogram. By engaging students in a routine that involved Pre-Proof activities focused on claims and, where applicable, justifications, Shane provided students with opportunities to explore or “experience” mathematical objects (Schoenfeld, 1986) and develop conjectures prior to working on proofs about those objects.

During-the-Proof Claims and Justifications. Four codes were developed for Claims and Justifications made During the Proof. Across the 21 theorems, we identified 60 student-generated claims made during the proof. The first code, which was related to stating the proof assumption and justifying it by “Given” only occurred once. We hypothesize that this aspect of proving needed to be carried out only once so that students would understand this proof requirement. The next two activity codes were similar to activities that occurred during the Pre-Proof. There were 11 instances of students stating claims that were conclusions drawn directly from the “Given” statement. The justification for such claims was typically the definition of the mathematical object that was the subject of the theorem (e.g., definition of parallelogram), but, at times, it was also appropriate to use a theorem about the mathematical object to justify a claim made directly from the “Given” statement. Also, similar to an activity described above, there were six claims generated through combinations of a postulate and an appropriate assumption that could be made about the diagram. The majority of student-generated claims (n=41) were related to the statements and reasons that followed once the initial conclusions were drawn from the hypothesis and any valid, relevant assumptions made about the diagram were identified.

Setting up the Proof

Setting up the Proof involved a range of activities including: working with the theorem as a conditional statement, developing the “Given” and “Prove” statements, and developing or working with a diagram for the proof. During the Setting-up-the-Proof activities, Shane attended to different aspects of setting up the proof, working on different competencies across the 21 theorems, over time. For example, for 8 of the 21 theorems, rather than providing students with the conditional statements of the theorems to be proved, Shane drew from the conjectures students developed during the Pre-Proof activities. Since these conjectures were often written using “everyday language,” such as “Opposite sides of a parallelogram are congruent,” when Setting up the Proof, Shane led discussions that supported his students to identify the assumptions (or hypotheses) in the conjecture (e.g., [If] a quadrilateral is a parallelogram) as well as the conclusions of the conjecture (e.g., [then] the opposite sides are congruent). For 11 of 21 theorems, students were not provided with the “Given” and “Prove” statements but rather had to participate in developing them during the whole-class discussions. For 6 of 11 of these theorems, students also played a role in generating the particular figures that would be used in the proof. Six of the 21 theorems proved during the observations were converses of other theorems that the class had also proved. Thus, it is unsurprising that discussions about the truth values of the converses of six of the theorems occurred. Last, for 12 of 21 of the theorems, a figure was provided for the students, but it was not marked. For example, for parallelogram proofs, Shane had pre-populated parallelograms labeled ABCD on his advanced organizer, but for each of the theorems, the diagrams still needed to be marked to reflect what students knew to be true from what they determined to be the “Given” information. Across the 21 theorems, by modifying what information was provided and what Shane left blank for the students to develop, Shane provided students with opportunities to develop different competencies needed to set up the proofs.

Concluding the Proof

Across the 21 theorem-proofs, Shane’s facilitation of Concluding the Proof activities included three noteworthy features. For 9 of the 16 theorems that did not have “names” such as The Midpoint Theorem or the Base Angles Theorem, Shane concluded the proofs by developing a shorthand version of the theorem that students could use moving forward. For example, Shane suggested that students could write "⊥ lines → ≅ Adjacent ∠s" rather than writing out the full text of the theorem: "If the two lines are perpendicular, then they form congruent adjacent angles." Referring to the shorthand notation, Shane stated, "Your options are either to write something like this, or you may just write the whole thing." Second, for 12 of the 21 theorems, Shane restated or rephrased the theorem after the class proved it, typically in a way that seemed intended to foster an understanding of what the class had just proved. For example, after proving the converse of the Isosceles Triangle or Base Angles Theorem, Shane stated: "So if you do have a pair of angles that are congruent in the triangle, it does imply that the sides opposite them are congruent, which implies it is an isosceles triangle." Last, upon completing 9 of the 21 theorem-proofs, Shane explained to students or reminded them that once a theorem was proven, it could be used in future proofs. For example, after writing shorthand notation for the third of four parallelogram theorems that they would prove that day, Shane asked his students, "So now we have how many properties of parallelograms we can use?" After establishing that they had three theorems plus the definition of parallelogram, Shane asked students to prove the fourth theorem of the day and reminded them: "Remember now, we, you can use any properties that we have already proved. So now you can use everything except for the one you're trying to prove." In doing so, Shane established that once a theorem was proved it could be used to prove other theorems; he also reminded the students not to engage in circular reasoning.

Proof Assessment Results

As explained in the Methods section, we calculated results for two of the full-proof items from Senk's (1983) CDASSG assessment that were administered in this study. For two groups of students - Shane's Grade 8 Honors students and Grade 8 Honors students who had teachers other than Shane (i.e., "non-Shane") - we calculated the numbers and percentages of students who were "Successful" (i.e., scored at least a 3 out of 4 points) on *both* Items 4 and 5, on *either* Item 4 or 5 but not both, and on *neither* Item 4 nor Item 5. As can be seen in Table 1, there were large differences between the results of the two groups of students. Acknowledging that the student populations for the two studies differed, as another point of comparison, in Senk's (1983) study, approximately 43% of students were successful on Item 4 and approximately 37% of students were successful on Item 5. Percentages of success for the same items in our study were 77% and 84% for Shane's students, respectively, and 30% and 25% for non-Shane students, respectively. This is noteworthy given that the students in this sub-study were all Grade 8 Honors students, whereas the students in Senk's study included a population of Honors and non-Honors students.

Table 1: Student Assessment Results for Shane's Students Compared to Other (Non-Shane) Grade 8 Honors Students

	Number (%) of Students ***Successful on both 4 & 5***	**Number (%) of Students** ***Successful on either 4 or 5, but not both***	**Number (%) of Students** ***Successful on neither 4 nor 5***
Shane's Students (n=128)	83 (64.8%)	40 (31.3%)	5 (3.9%)
Non-Shane's Students (n=336)	53 (15.8%)	78 (23.2%)	205 (61.0%)

Discussion and Conclusions

As noted by Herbst and Miyakawa (2008), while all theorems have proofs, in U.S. geometry classrooms, not every theorem is proved. The study reported here is significant in that it describes a routine for proving theorems - an activity that is apparently lacking in many U.S. classrooms. Furthermore, our assessment results provide evidence that the strategies employed by Shane seemed to be reasonably effective given that nearly two-thirds of Shane's students were successful on the two full-proof tasks analyzed for this study. It is interesting to note that the percentage of Shane's students who *were successful* on both proof items analyzed (about 65%) is very close to the percentage of students from non-Shane classrooms who *were not successful* on either proof task (61%). One limitation of this study, however, is that due to space constraints, we did not report more sophisticated statistical analyses controlling for various factors, and we did not determine statistical significance when taking these factors into account.

In contrast to reports by Cirillo and colleagues (2017), who noted that proof is often taught in a show-and-tell manner, we saw evidence that, in Shane's classroom, students were expected to make claims and provide justifications for their claims. This was evident in the way that Shane continuously asked students questions about what they *believed* to be true during the 21 theorem-proof episodes. Summing together codes from the Pre-Proof and During-Proof activities, it is also noteworthy that, within the data set of 21 theorem-proofs, we identified a total of 19 instances of students drawing conclusions directly from the hypothesis of the theorem. This is important because drawing valid conclusions from the proof assumptions has been identified as an important competency in proving, particularly for *beginning* a chain of reasoning, a skill in which many students struggle (Senk, 1985; Cirillo and Hummer, 2019, 2021). Also, there were 8 instances, in total, of students generating claims through a combination of a postulate and an assumption about the diagram. Cirillo and Hummer (2019) pointed out that making valid assumptions about diagrams is an under-recognized, but important proof competency.

Across the three features Shane incorporated during the Concluding the Proof activity, one important take-away is that these activities often seemed to accomplish what Herbst and Miyakawa (2008) identified as "sanctioning" the theorem, which involves explicitly declaring it as having that label. Shane sanctioned theorems by restating them, establishing shorthand notation for writing them, and acknowledging that they could now be used in future proofs.

Herbst and colleagues (2006, 2009) provided evidence which suggests that teachers heavily control the work of proving in American classrooms. Although, as evidenced by the data, through his question-and-answer exchanges to support students' development of claims, and through the ways Shane scaffolded the Setting up the Proof activities by alternating which competencies students had opportunities to practice while proving any one theorem, Shane did seem to provide students with more opportunities to authentically engage in proving theorems than research suggests is typical. To start, in contrast to the claim made by Herbst and Brach (2006), that students were not expected to prove theorems, Shane *did* expect his students, not only to prove theorems, but to heavily participate in proving them. For numerous theorems, Shane also expected students to participate in sketching their own diagrams and in determining the "Given" and "Prove" statements from the conjecture or the conditional statement. Thus, in contrast to the teachers from Herbst and colleagues' (2009) study, Shane did, in fact, seem to expect his students to carry a good deal of the cognitive load. To be clear, we are not suggesting that we disagree with the norms put forth by Herbst and colleagues. Rather, we mention these norms to demonstrate that Shane's approach seems to be unusual, and, given his students' test results, is worthy of examination. One question that this study raises is related to how effective Shane's teaching approach would be with a non-Honors student population. In other words, would Shane's approach work well for heterogeneous groups of more "typical" students?

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